AN EXTENSION OF THE GEOMETRICAL FORM OF THE HAHN-BANACH THEOREM

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The geometrical form of the Hahn-Banach theorem is well known (See for example [1]). The purpose of this note is to generalize this result to convergence vector space. More precisely, we shall prove the following

Theorem. *Let (E, τ) be a Hausdorff convergence vector space, let M be an affine subspace in E, and let K be a non-empty convex open subset of E, not intersecting M. There exists a closed hyperplane in E, containing M and not intersecting K.*

All vector spaces in this note are defined on the field $\boldsymbol{K}$ of real or complex numbers. We shall make use of the definitions and the notions of [3]. However we will recall some basic definitions for easy reference. A convergence structure on a set E is a mapping τ of E into the power set of the set of all filters on E satisfying the following conditions for each $x \in E$: (1) The filter $\dot{x}$ with base $\{\{x\}\}$ is in τx; (2) If $\mathfrak{G}$ is a filter on E containing a filter $\mathfrak{F} \in \tau x$, then $\mathfrak{G} \ni \tau x$; (3) If $\mathfrak{F}, \mathfrak{G} \in \tau x$, then $\mathfrak{F} \cap \mathfrak{G} \in \tau x$. It is said to be Hausdorff if $\tau x \cap \tau y \neq \phi$ implies $x = y$. An ordered pair (E, τ) of a vector space E and a covergence structure τ on E is called a convergence vector space if the mapping $(x, y) \longrightarrow x + y$ of $E \times E$ into E and the mapping $(\lambda, x) \longrightarrow \lambda x$ of $\boldsymbol{K} \times E$ into E are continuous with respect to τ and the usual topology on $\boldsymbol{K}$. It is easy to see that if (E, τ) is a convergence vector space, then $\lambda \cdot \tau 0 \subset \tau 0$ for every $\lambda \in \boldsymbol{K}$ and $\boldsymbol{V} \cdot x \in \tau 0$ for every $x \in E$, where $\boldsymbol{V}$ denotes the neighborhood filter of $0 \in \boldsymbol{K}$ for the usual topology on $\boldsymbol{K}$.

To prove the Theorem we recall several results.

Lemma 1. (Wloka [4]). *Let M be a vector subspace of a convergence vector space (E, τ). Then M is closed if and only if $(E/M, \tau')$ is Hausdorff, where τ' is the finest convergence structure on E/M which makes the canonical mapping of E onto E/M continuous.*

Lemma 2. (Courant [2]). *Every Hausdorff convergence vector space of finite dimension n is isomorphic to $\boldsymbol{K}^n$ with the usual topology.*

Then we have the following

Lemma 3. *Let (E, τ) be a Hausdorff convergence vector space over $\boldsymbol{R}$ of dimension at least 2. If K is an open, convex subset of E not containing 0, there exists a one-dimensional subspace of E not intersecting K.*

Proof. Let M be any fixed two dimensional subspace of E. If $M \cap K = \phi$, the result is immediate. Therefore we assume that $K_1 = M \cap K$ is non-empty. We see that K_1 is an open convex subset of M not containing 0. Because, for every $x \in K_1$, there is a $V \in \tau 0$ such that $(x+V) \cap M \subset (K \cap M)$. Therefore K_1 is open.

By Lemma 2 we can identify M with $\boldsymbol{R}^2$ (the Euclidean plane). Project K_1 onto a subset of the unit circle C of M by the mapping

$$f\colon (x,y) \longrightarrow \left(\frac{x}{r}, \frac{y}{r}\right), \qquad r = (x^2+y^2)^{\frac{1}{2}}.$$

Since K_1, being convex, is connected, $f(K_1)$ is connected, for f is continuous on K_1. Moreover $f(K_1)$ is an open subset of C. Hence $f(K_1)$ is an open arc on C which subtends an angle $\leqslant \pi$ at 0. Otherwise, there would exists a straight line in M passing through 0 and not intersecting K. This completes the proof.

To prove our Theorem we can assume that M is a subspace of E. Because, after a translation, if necessary, we can have $0 \in M$. Consider the family $\mathfrak{M}$ of all closed real subspaces of E that contain M and do not intersect K. Since K is open, $\overline{M}$ is the closed subspace such that $M \cap K = \phi$. Hence $\mathfrak{M}$ is non-empty. Order $\mathfrak{M}$ by inclusion $\supset$. If we have a totally ordered subfamily $\{M_\alpha\}$ of $\mathfrak{M}$, the closure of $\cup^\alpha M_\alpha$ is clearly its least upper bound. Thus Zorn's lemma applies and we may conclude that $\{M_\alpha\}$ possesses maximal element H_0. If E_0 denotes the real underlying space of E, the quotient space E_0/H_0 is Hausdorff by Lemma 1, for H_0 is closed. Because of $K = \phi$, E_0/H_0 has dimension $\geqq 1$. Suppose that E_0/H_0 is of dimension $\geqq 2$. Since the natural mapping Φ of E_0 onto E_0/H_0 is linear open, $G = \Phi(K)$ is a convex, open subset of E_0/H_0, not containing 0, since H_0 does not intersect K. Hence by Lemma 3, there exists a one-dimensional subspace N of E_0/H_0 not intersecting G. This implies that $H = \Phi^{-1}(N)$ is a closed subspace of E_0 containing H_0 properly and not intersecting K. This contradicts thè maximality of H_0 in $\mathfrak{M}$. Hence E_0/H_0 has dimension 1, and H_0 is a closed, real hyperplane containing M and not intersecting K. This completes the proof when E is a convergence vector space over $\boldsymbol{R}$.

E is a convergence vector space over C, then $M = iM$ (assuming $0 \in M$), since M is a subspace of E. Consequently $H_1 = H_0 \cap iH_0$, which is a closed hyperplane in E not intersecting K, contains M, and the proof is complete.

Corollary. *If E is a Hausdorff convergence vector space, there exists a continuous linear form $f \neq 0$ on E if and only if E cotains a non-empty convex, open subset $K \neq E$.*

Proof. If $f \neq 0$ is a continuous linear form on E, the subset $K = \{x : |f(x)| \leqq 1\}$ is $\neq E$, convex, and open. Conversely, if the convex set $K \subset E$ is open and $x_0 \in K$, x_0 is a closed hyperplane H (not intersecting K) by Theorem. Since E/H is a one-dimensional subspace, it follows from Lemma 2 that there exists a non-zero continuous linear form of E. This completes the proof.

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